PROBLEMA 1.

 *Să se determine funcțiile* $f:N^{\*}\rightarrow R $*cu proprietatea :*

$$f\left(1\right)+2∙f\left(2\right)+3∙f\left(3\right)+…+n∙f\left(n\right)=f\left(n+1\right)-1, ∀ n\in N^{\*}$$

Soluție.

Pentru n=1 relația devine $f\left(2\right)=1+f\left(1\right)$. (1 punct )

Pentru n=2 relația devine $f\left(1\right)+2f\left(2\right)=f\left(3\right)-1⟺f\left(3\right)=3\left(1+f\left(1\right)\right).$ ( 1 punct )

Pentru n=3 relația dată conduce la $f\left(4\right)=12∙\left(1+f\left(1\right)\right)$. (1 punct )

Prin inducție se arată că $f\left(n\right)=\frac{n!}{2}\left(1+f\left(1\right)\right)$, pentru orice $n\geq 2$. (unde $ n!=1∙2∙…∙n)$

 (3 puncte)

Notând f(1)=a, unde a este un număr real, rezultă că funcțiile căutate sunt:

$f: N^{\*}\rightarrow R, f\left(n\right)=\left\{\begin{array}{c}a, \&n=1\\\frac{n!\left(a+1\right)}{2}, \&n\geq 2\end{array}\right.$ *.* ( 1 punct )

PROBLEMA 2.

Să se rezolve în $R$ ecuația:

$$\sqrt{x^{2}+31x}+ \sqrt{x+31} =x+ \sqrt{x}+8$$

Soluție:

|  |  |
| --- | --- |
| $$\sqrt{x(x+31)}+ \sqrt{x+31} =\sqrt{x}(\sqrt{x}+ 1)+8$$$$\sqrt{x+31 } (\sqrt{x}+ 1) =\sqrt{x}(\sqrt{x}+ 1)+8$$$$(\sqrt{x}+1) \left(\sqrt{x+31 } - \sqrt{x}\right)=8$$ | 2p |
|  $ notăm a= \sqrt{x+31 } și b= \sqrt{x} $$\left\{\begin{array}{c}\left(b+1\right)\left(a-b\right)=8\\a^{2} - b^{2} = 31 \end{array} \right. ⟺\left\{\begin{array}{c}\left(b+1\right)\left(a-b\right)=8\\\left(a-b\right)\left(a+b\right)=31\end{array} \right.$  | 2p |
| $$dar a\ne b ⇒ \frac{b+1}{a+b}= \frac{8}{31} ⟺a= \frac{23b+31}{8} ⇒15b^{2}+46b-33=0$$ | 2p |
| $$b\_{1}= - \frac{11}{3}, b\_{2}= \frac{3}{5} ⟹x=\frac{9}{25}$$ | 1p |

PROBLEMA 3.

Fie rombul *ABCD*  și punctele $M \in \left(AB\right), N \in \left(BC\right), P \in \left(CD\right).$ Să se arate că centrul de greutate al triunghiului *MNP* aparține dreptei *AC* dacă și numai dacă *AM + DP = BN.*

Soluție:

Fie *R* mijlocullui $\left[NP\right] iar MR \bigcap\_{}^{}AC= \left\{G\right\}$

$$Considerăm \frac{MG}{GR}=k, AB=a, ……………………………………………………………………. 1p$$

$$\vec{AG}=\frac{1}{1+k} \vec{AM}+\frac{k}{1+k}\vec{AR} =\frac{1}{1+k}\frac{AM}{a}\vec{AB}+ \frac{k}{1+k} \frac{1}{2} \left(\vec{AN}+ \vec{AP}\right) $$

$$=\frac{1}{1+k}\frac{AM}{a}\vec{AB}+ \frac{k}{2(1+k)} \left(\vec{AB}+ \vec{BN}\right)+ \frac{k}{2(1+k)} \left(\vec{AD}+ \vec{DP}\right) $$

$$=\frac{AM}{\left(1+k\right)a}\vec{AB}+ \frac{k}{2(1+k)} \vec{AB}+ \frac{k}{2(1+k)} \frac{BN}{a} \vec{BC} + \frac{k}{2(1+k)} \vec{AD}+ \frac{k}{2(1+k)} \frac{DP}{a} \vec{DC}$$

$$=\left(\frac{AM}{\left(1+k\right)a}+ \frac{k}{2(1+k)}+ \frac{kDP}{2a(1+k)}\right)∙ \vec{AB}+ \left(\frac{kBN}{2a\left(1+k\right)}+ \frac{k}{2\left(1+k\right)}\right)∙ \vec{AD}………..2p $$

$$\vec{AC}= \vec{AB}+\vec{AD}$$

$$\vec{AG}, \vec{AC} coliniari ⇔ \frac{AM}{\left(1+k\right)a}+ \frac{k}{2(1+k)}+ \frac{kDP}{2a(1+k)}= \frac{kBN}{2a(1+k)}+\frac{k}{2(1+k)}$$

$$\frac{AM}{\left(1+k\right)a}+ \frac{kDP}{2a(1+k)}= \frac{kBN}{2a(1+k)} $$

$$AM+\frac{k}{2}DP= \frac{k}{2}BN \left(\*\right) …………………………………………………………………………. 3p$$

$G centru de greutate ⟺ k=2 ⇔AM+DP = BN$ …………………………………..*1p*

PROBLEMA 4.

1. Se dau numerele$ x,y,z>0$ pentru care $x+y+z=2$.
2. Să se demonstreze că $\frac{x-y}{xy+2z}+\frac{y-z}{yz+2x}+\frac{z-x}{zx+2y}=0.$
3. Să se demonstreze că $\frac{x}{xy+2z}+\frac{y}{yz+2x}+\frac{z}{zx+2y}\geq \frac{9}{8}$.

Barem

$\frac{x-y}{xy+2z}+\frac{y-z}{yz+2x}+\frac{z-x}{zx+3y}=\frac{x-y}{xy+\left(x+y+z\right)z}+\frac{y-z}{yz+\left(x+y+z\right)x}+\frac{z-x}{zx+\left(x+y+z\right)y}=\frac{x-y}{\left(x+z\right)\left(y+z\right)}+\frac{y-z}{\left(x+y\right)\left(x+z\right)}+\frac{z-x}{\left(x+y\right)\left(y+z\right)}=\frac{x^{2}-y^{2}+y^{2}-z^{2}+z^{2}-x^{2}}{\left(x+y\right)\left(x+z\right)\left(y+z\right)}=0.$ ............................**3p**

1. $\frac{x}{xy+2z}+\frac{y}{yz+2x}+\frac{z}{zx+2y}=\frac{y}{xy+2z}+\frac{z}{yz+2x}+\frac{x}{zx+2y}⟹\frac{x}{xy+2z}+\frac{y}{yz+2x}+\frac{z}{zx+2y}=\frac{1}{2}\left(\frac{x+y}{\left(x+z\right)\left(y+z\right)}+\frac{y+z}{\left(x+y\right)\left(x+z\right)}+\frac{z+x}{\left(x+y\right)\left(y+z\right)}\right)$ (1)..........................................**1p**

Notăm. $x+y=a,y+z=b, z+x=c , a,b,c>0$

$\frac{1}{2}\left(\frac{a}{bc}+\frac{b}{ac}+\frac{c}{ab}\right)=\frac{1}{2}∙\frac{a^{2}+b^{2}+c^{2}}{abc}\geq \frac{1}{2}∙\frac{ab+ac+bc}{abc}=\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$ (2)............................**1p**

$\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3}\geq \frac{3}{a+b+c}⟹\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq \frac{9}{4}$ (3)............................**1p**

$⇒\frac{x}{xy+2z}+\frac{y}{yz+2x}+\frac{z}{zx+2y}\geq \frac{9}{8}$ .............................**1p**